Recurrence Relations for Single and Product Moments of Generalized Order Statistics from Generalized Power Weibull Distribution and its Characterization

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Abstract: Generalized power Weibull distribution was firstly discussed by Nikulin and Haghighi (2006). In this article recurrence relations for single and product moments of generalized order statistics (*gos*) for Generalized power Weibull distribution have been driven. Moments of order statistics and k-upper records are discussed as special cases. Characterization of generalized power Weibull distribution through conditional expectation is also presented.

Keywords: Generalized order statistics, upper-k record, single moments, product moments, recurrence relations, generalized power Weibull distribution, characterization, conditional expectation.
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1. INTRODUCTION

Let $X_1, X_2,...,X_n$ be a sequence of independent and identically distributed (*iid*) random variables (*rv*) with absolutely continuous distribution function (*df*) F(x) and the probability density function (*pdf*) f(x), $x \in (\alpha,\beta)$. Let $n \in N, (n \geq 2),\ k \geq 1,\ \tilde{m} = (m_1, ..., m_{n-1}) \in R^{n-1},\ M_r = \sum_{j=r}^{n-1} m_j,\ 1 \leq r \leq n-1$ be the parameters such that $\gamma_r = k+n-r+M_r > 0$, for all $r \in \{1, 2,.., n-1\}$. Then $X(r,n,\tilde{m},k)$ are called generalized order statistics (*gos*) if their joint *pdf* is given by

$$f_{X(1,n,\tilde{m},k),...,X(n,n,\tilde{m},k)}(x_1, x_2,..., x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)(1-F(x_n))^{k-1} f(x_n)$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq ... \leq x_n < F^{-1}(1)$.

Choosing the parameters appropriately, models such as ordinary order statistics ($\gamma_i = n-i+1$; $i=1, 2,...,n$, i.e. $m_1 = m_2 =...= m_{n-1} = 0$, $k=1$), k^{th} record values ($\gamma_i = k$, i.e. $m_1 = m_2 =...= m_{n-1} = -1$, $k \in N$), sequential order statistics ($\gamma_i = (n-i+1)\alpha_i$; $\alpha_1,\alpha_2,...,\alpha_n > 0$), order statistics with non-integral sample size ($\gamma_i = \alpha-i+1$, $\alpha > 0$). Pfeifer's record values ($\gamma_i = \beta_i$; $\beta_1,\beta_2,...,\beta_n > 0$) and Progressive type II censored order statistics ($m_i \in N_0$, $k \in N$) are obtained (Kamps (1995a, 1995b), Kamps and Cramer (2001)).

For $\gamma_i \neq \gamma_j$, $i \neq j$ for all $i, j \in (1, 2,...,n)$, the *pdf* of $X(r,n,\tilde{m},k)$ is given by Kamps and Cramer (2001) in the following way

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1} f(x)\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i-1}, \quad -\infty < x < \infty. \tag{1}$$

The joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \leq r < s \leq n$ is given as

$$f_{X(r,n,\tilde{m},k),\ X(s,n,\tilde{m},k)}(x,y) = c_{s-1}\left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i}\right) . \left(\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right)\frac{f(x)}{[1-F(x)]}\frac{f(y)}{[1-F(y)]}, \tag{2}$$

where $-\infty < x < y < \infty$ and

$$a_i(r) = \prod_{\substack{j=1 \\ j\neq i}}^{r}\frac{1}{(\gamma_j - \gamma_i)},\ 1 \leq i \leq r \leq n,$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j\neq i}}^{s}\frac{1}{(\gamma_j - \gamma_i)},\ r+1 \leq i \leq s \leq n.$$

It may be noted that for $m_1 = m_2 = ... = m_n \neq 1$,

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{r-i}, \qquad (3)$$

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-i}. \qquad (4)$$

Therefore *pdf* of $X(r,n,\tilde{m},k)$ given in (1) reduces to

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[1-F(x)]^{\gamma_r -1} f(x) g_m^{r-1}(F(x)), \qquad (5)$$
$$-\infty < x < \infty,$$

and joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$ given in (2) reduces to

$$f_{X(r,n,m,k),\, X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}[1-F(x)]^{m}$$
$$.f(x) g_m^{r-1}(F(x))[h_m(F(y))-h_m(F(x))]^{s-r-1}$$
$$.[1-F(y)]^{\gamma_s -1} f(y), \quad -\infty < x < y < \infty, \qquad (6)$$

where

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1),$$

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \; x \in (0, 1).$$

A random variable X is said to have the generalized power Weibull distribution if its *pdf* is of the form

$$f(x) = \alpha\theta x^{\alpha-1}(1+x^{\alpha})^{\theta-1}\exp(1-(1+x^{\alpha})^{\theta}), \qquad (7)$$
$$x \geq 0, \; \alpha>0, \; \theta>0$$

and corresponding *df*

$$F(x) = 1-\exp(1-(1+x^{\alpha})^{\theta}) \quad x \geq 0, \; \alpha,\theta>0. \qquad (8)$$

Now in view of (7) and (8), we get

$$f(x) = \alpha\theta \sum_{b=0}^{\theta-1} \binom{\theta-1}{b} x^{\alpha b+\alpha-1}[1-F(x)]. \qquad (9)$$

The relation (9) will be utilized to establish recurrence relations for moments of *gos*.

2. SINGLE MOMENTS

Theorem 2.1. For the generalized power Weibull distribution as given in (7) and $n \in N, \; \tilde{m} \in R, \; k>0, \; 1 \leq r \leq n, \; p=1,2,...$

$$E[X^p(r,n,\tilde{m},k)] = \alpha\theta\gamma_r \sum_{b=0}^{\theta-1} \binom{\theta-1}{b} \left(\frac{1}{\alpha b+\alpha+p}\right)$$
$$.\left\{E[X^{\alpha b+\alpha+p}(r,n,\tilde{m},k)] - E[X^{\alpha b+\alpha+p}(r-1,n,\tilde{m},k)]\right\}. \qquad (10)$$

Proof. Using (1) and (9), we get

$$E[X^p(r,n,\tilde{m},k)] = \alpha\theta \sum_{b=0}^{\theta-1} \binom{\theta-1}{b}$$
$$.c_{r-1}\int_0^{\infty} x^{\alpha b+\alpha+p-1}\left(\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right)dx. \qquad (11)$$

Integrating by parts (11) treating $x^{\alpha b+\alpha+p-1}$ for integration and rest of the integrand for differentiation we have

$$E[X^p(r,n,\tilde{m},k)] = \alpha\theta\gamma_r \sum_{b=0}^{\theta-1} \binom{\theta-1}{b} \left(\frac{1}{\alpha b+\alpha+p}\right)$$
$$.c_{r-1}\int_0^{\infty} x^{\alpha b+\alpha+p}\left(\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i -1}\right) f(x)\, dx$$
$$-\alpha\theta \sum_{b=0}^{\theta-1} \binom{\theta-1}{b} \left(\frac{1}{\alpha b+\alpha+p}\right) c_{r-1}$$
$$.\int_0^{\infty} x^{\alpha b+\alpha+p}\left(\sum_{i=1}^{r-1} a_i(r-1)(1-F(x))^{\gamma_i -1}\right) f(x)\, dx,$$

which after simplification yields equation (10).

Corollary2.1. For $m_1 = m_2 = ... = m_{n-1} = m \neq -1$, the recurrence relation for single moments of *gos* for generalized power Weibull distribution is given as

$$E[X^p(r,n,m,k)] = \alpha\theta\gamma_r \sum_{b=0}^{\theta-1} \binom{\theta-1}{b} \left(\frac{1}{\alpha b+\alpha+p}\right)$$
$$.\left\{E[X^{\alpha b+\alpha+p}(r,n,m,k)] - E[X^{\alpha b+\alpha+p}(r-1,n,m,k)]\right\}.$$

Proof. This can easily be deduced from (10) in view of the relation (3).

Remark 2.1. Recurrence relation for single moments of order statistics (at m=0, k=1) is

$$E[X_{r:n}^p] = \alpha\theta(n-r+1) \sum_{b=0}^{\theta-1} \binom{\theta-1}{b} \left(\frac{1}{\alpha b+\alpha+p}\right)$$
$$.\left\{E[X_{r:n}^{\alpha b+\alpha+p}] - E[X_{r-1:n}^{\alpha b+\alpha+p}]\right\}.$$

Remark 2.2. Recurrence relation for single moments of k-th upper record (at m= -1) will be

$$E(X_r^{(k)})^p = \alpha\theta k \sum_{b=0}^{\theta-1} \binom{\theta-1}{b} \left(\frac{1}{\alpha b+\alpha+p}\right)$$
$$.\left\{E(X_r^{(k)})^{\alpha b+\alpha+p} - E(X_{r-1}^{(k)})^{\alpha b+\alpha+p}\right\}.$$

3. PRODUCT MOMENTS

Theorem 3.1. For distribution given in (7). Fix a positive integer k and for $n \in N,\ \tilde{m} \in R,\ 1 \le r < s \le n,$

$$E[X^p(r,n,\tilde{m},k)X^q(s,n,\tilde{m},k)]=\alpha\theta\gamma_s\sum_{b=0}^{\theta-1}\binom{\theta-1}{b}\cdot\left(\frac{1}{\alpha b+\alpha+q}\right)\Big\{E[X^p(r,n,\tilde{m},k)X^{\alpha b+\alpha+q}(s,n,\tilde{m},k)]-E[X^p(r,n,\tilde{m},k)X^{\alpha b+\alpha+q}(s-1,n,\tilde{m},k)]\Big\}. \quad (12)$$

Proof: Using (2) and (9), we have

$$E[X^p(r,n,\tilde{m},k)X^q(s,n,\tilde{m},k)]=c_{s-1}\int_0^\infty x^p \cdot\left(\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}\right)\frac{f(x)}{[1-F(x)]}I(x)dx, \quad (13)$$

where

$$I(x)=\alpha\theta\sum_{b=0}^{\theta-1}\binom{\theta-1}{b}\int_x^\infty y^{\alpha b+\alpha+q-1}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i}\right)dy.$$

Integrating I(x) by parts, we get

$$I(x)=\alpha\theta\gamma_s\sum_{b=0}^{\theta-1}\binom{\theta-1}{b}\left(\frac{1}{\alpha b+\alpha+q}\right)\int_x^\infty y^{\alpha b+\alpha+q}\cdot\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i}\right)\frac{f(y)}{[1-F(y)]}dy$$
$$-\alpha\theta\sum_{b=0}^{\theta-1}\binom{\theta-1}{b}\left(\frac{1}{\alpha b+\alpha+q}\right)\int_x^\infty y^{\alpha b+\alpha+q}\cdot\left(\sum_{i=r+1}^{s-1}a_i^{(r)}(s-1)\left[\frac{1-F(y)}{1-F(x)}\right]^{\gamma_i}\right)\frac{f(y)}{[1-F(y)]}dy.$$

Substituting the above expression into equation (13) and simplifying the resulting equation, we get equation (12).

Corollary 3.1. For $m_1=m_2=...=m_{n-1}=m\neq -1,$ the recurrence relation for product moments is given as

$$E[X^p(r,n,m,k)X^q(s,n,m,k)]=\alpha\theta\gamma_s\sum_{b=0}^{\theta-1}\binom{\theta-1}{b}\cdot\left(\frac{1}{\alpha b+\alpha+q}\right)\Big\{E[X^p(r,n,m,k)X^{\alpha b+\alpha+q}(s,n,m,k)]-E[X^p(r,n,m,k)X^{\alpha b+\alpha+q}(s-1,n,m,k)]\Big\}. \quad (14)$$

Proof. Equation (14) can be reduced from (12) in view of (3) and (4) or by replacing $\tilde{m}$ with m in (12).

Remark 3.1. Recurrence relation for product moments of order statistics (at m = 0, k =1) is

$$E[X_{r:n}^p X_{s:n}^q]=\alpha\theta(n-s+1)\sum_{b=0}^{\theta-1}\binom{\theta-1}{b}\left(\frac{1}{\alpha b+\alpha+q}\right)\Big\{E[X_{r:n}^p X_{s:n}^{\alpha b+\alpha+q}]-E[X_{r:n}^p X_{s-1:n}^{\alpha b+\alpha+q}]\Big\}.$$

Remark 3.2. Recurrence relation for product moments of k-th upper record values will be

$$E[(X_r^{(k)})^p(X_s^{(k)})^q]=\alpha\theta k\sum_{b=0}^{\theta-1}\binom{\theta-1}{b}\left(\frac{1}{\alpha b+\alpha+q}\right)\cdot\Big\{E[(X_r^{(k)})^p(X_s^{(k)})^{\alpha b+\alpha+q}]-E[(X_r^{(k)})^p(X_{s-1}^{(k)})^{\alpha b+\alpha+q}]\Big\}.$$

Remark 3.3. At p=0, we obtain recurrence relation for single moments as given in (10).

4. CHARACTERIZATION

Let $X(r,n,m,k)$, r =1, 2,… be *gos,* then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k)=x,\ 1\le r<s\le n$, in view of (5) and (6) is

$$f_{s|r}(y|x)=\frac{c_{s-1}}{(s-r-1)!c_{r-1}}[h_m(F(y))-h_m(F(x))]^{s-r-1}\cdot\frac{[1-F(y)]^{\gamma_s-1}}{[1-F(x)]^{\gamma_r-m-1}}f(y),\quad -\infty<x<y<\infty. \quad (15)$$

Theorem 4.1. Let X be an absolutely continuous *rv* with *df* F(x) and *pdf* f(x) with F(x)<1, for all $x\in(0,\infty)$. Then

$$F(x)=1-\exp(1-(1+x^\alpha)^\theta)\quad x\ge 0,\ \alpha,\theta>0, \quad (16)$$

if and only if

$$E\Big[\exp\big(-(1+x^\alpha(r+1,n,m,k))^\theta\big)|\ x(r,n,m,k)=x\Big]=\left(\frac{\gamma_{r+1}}{\gamma_{r+1}+1}\right)\exp\big(-(1+x^\alpha)^\theta\big). \quad (17)$$

Proof: From equation (15), for s = r + 1, we have

$$E\Big[\exp(-\lambda e^{x^\beta(r+1,n,m,k)})\ |\ x(r,n,m,k)=x\Big]=\frac{\gamma_{r+1}}{[1-F(x)]^{\gamma_{r+1}}}\int_x^\infty \exp(-\lambda e^{y^\beta})[1-F(y)]^{\gamma_{r+1}-1}f(y)\,dy. \quad (18)$$

After using (7) and calculating the integral, we obtain (16). To prove sufficient part, we have from (17) and (18)

$$E\Big[\exp\big(-(1+x^\alpha(r+1,n,m,k))^\theta\big)|\ x(r,n,m,k)=x\Big]=\frac{\gamma_{r+1}}{[1-F(x)]^{\gamma_{r+1}}}\int_x^\infty \exp\big(-(1+y^\alpha)^\theta\big)[1-F(y)]^{\gamma_{r+1}-1}f(y)\,dy.$$

Differentiating both the sides with respect to x and rearranging, we get $\frac{f(x)}{[1-F(x)]} = \alpha\theta x^{\alpha-1}(1+x^{\alpha})^{\theta-1},$ which leads to (16).

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